1. $\mathcal{L} \equiv \mathcal{L}_0 - \lambda \varphi_1^2 \varphi_2^2$

$$G^{(4)}(x_1,x_2,x_3,x_4) = \frac{1}{Z[0,0]} \int \mathcal{D}\varphi_1 \int \mathcal{D}\varphi_2 \, e^{i\int d^4x \, (\mathcal{L}_0 - \lambda \varphi_1^2 \varphi_2^2)} \varphi_1(x_1)\varphi_1(x_2)\varphi_2(x_3)\varphi_2(x_4)$$

$$Z[0,0] = \int \mathcal{D}\varphi_1 \int \mathcal{D}\varphi_2 \, e^{i\int d^4x \, \mathcal{L}_0}$$

The $O(\lambda)$ contributions to $G^{(4)}$ is given by:

$$G^{(4)}(x_1,\dots,x_4)\Big|_{O(\lambda)} = \frac{1}{Z[0,0]} \int \mathcal{D}\varphi_1 \int \mathcal{D}\varphi_2 \, e^{i\int d^4x \, \mathcal{L}_0} (-i\lambda) \int d^4w \, \varphi_1^2(w)\varphi_2^2(w)\varphi_1(x_1)\varphi_1(x_2)\varphi_2(x_3) \cdot \varphi_2(x_4)$$

$$= -i\lambda \int d^4w \, \langle \varphi_1^2(w)\varphi_2^2(w)\varphi_1(x_1)\varphi_1(x_2)\varphi_2(x_3)\varphi_2(x_4)\rangle$$

$$= -i\lambda \int d^4w \Big\{ \underset{x_1 w}{2} \cdot \underset{x_3 w}{2} \; iD_1(x_1-w)\, iD_1(x_2-w)\, iD_2(x_3-w)\, iD_2(x_4-w) + \text{+ disconnected contributions} \Big\}$$

The connected contribution gives the Feynman rule for the vertex after rewriting $iD_i(x_j-w)$ $i=1,2$ in terms of their Fourier transform and "amputating" the external legs

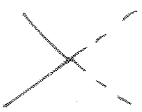
$$G_c^{(4)}(x_1,\dots,x_4)\Big|_{O(\lambda)} = -4i\lambda \int d^4w \, iD_1(x_1-w)\, iD_1(x_2-w)\, iD_2(x_3-w)\, iD_2(x_4-w)$$

$$= -4i\lambda \int d^4w \left(\prod_{i=1}^{4} \int \frac{d^4k_i}{(2\pi)^4} e^{-ik_i\cdot(x_i-w)} \right) \prod_{i=1}^{2} \frac{i}{k_i^2 - m_1^2 + i\varepsilon} \prod_{i=3}^{4} \frac{i}{k_i^2 - m_2^2 + i\varepsilon}$$

Use $\int d^4w \prod_{i=1}^{4} e^{ik_i\cdot w} = (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{4} k_i\right)$ to obtain

$$= \underbrace{\prod_{i=1}^{4} \int \frac{d^4k_i}{(2\pi)^4} e^{-ik_i\cdot x_i} \prod_{i=1}^{2} \frac{i}{k_i^2 - m_1^2 + i\varepsilon} \prod_{i=1}^{2} \frac{i}{k_i^2 - m_2^2 + i\varepsilon}}_{\text{external legs' factors}} (-4i\lambda)(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{4} k_i\right)$$

The Feynman rule in momentum space is then

 $-4i\lambda$

— φ_1
- - - - φ_2

($(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{4} k_i\right)$ for the conservation of total momentum at the vertex is implicit)

2. EoM for ψ, $\bar\psi$

2a) $\mathcal{L} = \bar\psi (i\overrightarrow{\partial\!\!\!/} - m)\psi$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} = \frac{\delta \mathcal{L}}{\delta \psi} \quad \text{gives} \quad -\overrightarrow{\partial}_\mu\, i\bar\psi\gamma^\mu = m\bar\psi$$

$$\bar\psi\,(i\overleftarrow{\partial\!\!\!/} + m) = 0$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar\psi} = \frac{\delta \mathcal{L}}{\delta \bar\psi} \quad \text{gives} \quad (i\overrightarrow{\partial\!\!\!/} - m)\psi = 0$$

2b) $$\partial_\mu J^\mu = \partial_\mu(\bar\psi\gamma^\mu\psi) = \bar\psi(\overleftarrow{\partial\!\!\!/} + \overrightarrow{\partial\!\!\!/})\psi \underset{\text{using EoM}}{=} \bar\psi(im - im)\psi = 0$$

3. $\mathcal{L} = \bar\psi(i\overrightarrow{\partial\!\!\!/} - m)\psi - \frac{G}{\sqrt{2}} J_\mu J^{\mu\dagger}$

3a) $D = dL - I$

L = # Loops
I = # Internal fermionic lines

$L = I - V + 1$ and the relation between the # vertices V and the number of external / internal lines is in this case

$4V = E + 2I$

V = # vertices 
E = # External fermionic lines

This gives

$$D = d(I - V + 1) - I = (d-1) I - d V + d$$

$$= (d-1) \frac{4V - E}{2} - dV + d$$

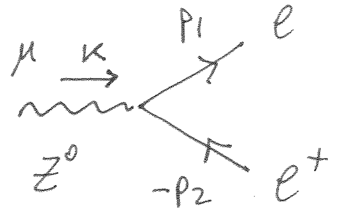
$$= d + (d-2) V - \left(\frac{d-1}{2}\right) E$$

3b) It is renormalizable in $d = 2$, where D does not depend on the number of vertices V.
This means that a finite number of 1PI <u>amplitudes</u> (not diagrams) will be superficially primitively UV divergent, i.e. $D \geq 0$
These amplitudes have $D = 2 - \frac{1}{2} E$ $(d = 2)$

$E = 0$ $D = 2$
$E = 2$ $D = 1$
$E = 4$ $D = 0$

NB E odd is excluded because no amplitude has an odd number of external fermionic lines.

4. $Z^0 \to e^+ e^-$



$$A = ig\,\bar{u}_r(\vec{p}_1)\,\gamma_\mu\,(1-\gamma_5)\,v_s(\vec{p}_2)\,\varepsilon^\mu_\alpha(\vec{k})$$ (Take ε^μ_α real)

$$A^\dagger = -ig\,v_s^\dagger(\vec{p}_2)\,(1-\gamma_5)\,\gamma_\mu^\dagger\,\gamma^0\,u_r(\vec{p}_1)\,\varepsilon^\mu_\alpha(\vec{k})$$

$\gamma^0 = \gamma^{0\dagger}$

$\gamma^{0^2} = \mathbb{1}$

$$= -ig\,\bar{v}_s(\vec{p}_2)\,\gamma^0(1-\gamma_5)\,\gamma_\mu^\dagger\,\gamma^0\,u_r(\vec{p}_1)\,\varepsilon^\mu_\alpha(\vec{k})$$

$$= -ig\,\bar{v}_s(\vec{p}_2)\,(1+\gamma_5)\,\gamma_\mu\,u_r(\vec{p}_1)\,\varepsilon^\mu_\alpha(\vec{k})$$

$\gamma^0\gamma_\mu^\dagger\gamma^0 = \gamma_\mu$

$$= -ig\,\bar{v}_s(\vec{p}_2)\,\gamma_\mu\,(1-\gamma_5)\,u_r(\vec{p}_1)\,\varepsilon^\mu_\alpha(\vec{k})$$

$$X = \frac{1}{3}\sum_{\alpha=1}^{3}\sum_{r,s=1}^{2} A A^\dagger$$

$$= \frac{1}{3}\sum_{\alpha=1}^{3}\sum_{r,s=1}^{2} g^2\left(\bar{u}_r(\vec{p}_1)\,\gamma_\mu(1-\gamma_5)\,v_s(\vec{p}_2)\right)\left(\bar{v}_s(\vec{p}_2)\,\gamma_\nu(1-\gamma_5)\,u_r(\vec{p}_1)\right)\varepsilon^\mu_\alpha(\vec{k})\,\varepsilon^\nu_\alpha(\vec{k})$$

$$= \frac{1}{3}g^2\,\mathrm{Tr}\left[\frac{\not{p}_1+m_e}{2m_e}\,\gamma_\mu\,(1-\gamma_5)\,\frac{\not{p}_2-m_e}{2m_e}\,\gamma_\nu\,(1-\gamma_5)\right]\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{M_Z^2}\right)$$

$(1-\gamma_5)^2 = 2(1-\gamma_5)$ $\quad\{\gamma_\mu,\gamma_5\}=0$

$(1-\gamma_5)(1+\gamma_5) = 0$ & $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha]=0$

imply that mass terms in the trace do not contribute

$$X = \frac{2}{3}g^2\frac{1}{(2m_e)^2}\,\mathrm{Tr}\left[\not{p}_1\,\gamma_\mu\,\not{p}_2\,\gamma_\nu\,(1-\gamma_5)\right]\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{M_Z^2}\right)$$

$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = -4i\varepsilon^{\mu\nu\alpha\beta}$ & $\varepsilon^{\mu\nu\alpha\beta}$ antisymmetric gives a vanishing contribution when contracted with the symmetric tensor $\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{M_Z^2}\right)$

$$X = \frac{g^2}{6 m_e^2} \, \mathrm{Tr}\left[\not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu\right] \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_Z^2}\right)$$

$$= \frac{2}{3} \frac{g^2}{m_e^2} \left(p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - g_{\mu\nu} \, p_1 \cdot p_2\right) \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_Z^2}\right)$$

$$= \frac{2}{3} \frac{g^2}{m_e^2} \left(p_1 p_2 + \frac{2 (k p_1)(k p_2)}{M_Z^2}\right)$$

4b) CM kinematics :

$k = p_1 + p_2$

Squaring gives $M_Z^2 = 2 m_e^2 + 2 p_1 \cdot p_2$ and $p_1 p_2 = \frac{M_Z^2}{2} - m_e^2$

$$k p_1 = p_1^2 + p_1 p_2 = m_e^2 + \frac{M_Z^2}{2} - m_e^2 = \frac{M_Z^2}{2}$$

$k p_2 = k p_1$

$$X = \frac{2}{3} \frac{g^2}{m_e^2} \left(M_Z^2 - m_e^2\right)$$ only dependent on m_e^2, M_Z^2

$$\Gamma = \frac{1}{16\pi^2} \frac{m_e^2}{M_Z} \sqrt{1 - \frac{4 m_e^2}{M_Z^2}} \cdot \frac{2}{3} \frac{g^2}{m_e^2} \left(M_Z^2 - m_e^2\right) \cdot 4\pi$$

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta$$

$$= \frac{g^2}{6\pi} M_Z \left(1 - \frac{m_e^2}{M_Z^2}\right) \sqrt{1 - \frac{4 m_e^2}{M_Z^2}}$$